

# Complexity of Weak, Strong and Dynamic Controllability of CNCUs

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## Abstract

A *Constraint Network Under Conditional Uncertainty (CNCU)* is a formalism able to model a constraint satisfaction problem (CSP) where variables and constraints are labeled by a conjunction of Boolean variables, or *booleans*, whose truth value assignments are *out of control* and only discovered upon the execution of their related *observation points* (special kind of variables). Before the execution of the CNCU starts (i.e., the *online* assignment of values to variables), we do not know completely which constraints and variables will be taken into consideration nor in which order. *Weak controllability* implies the existence of a strategy to execute a CNCU whenever the whole uncontrollable part is known before executing. *Strong controllability* is the opposite case and implies the existence of a strategy to execute a CNCU always the same way no matter how the uncontrollable part will behave. *Dynamic controllability* implies the existence of a strategy to execute a CNCU possibly differently depending on how the uncontrollable part is behaving. In this paper we prove that weak controllability is  $\Pi_2^P$ -complete, strong controllability is NP-complete and dynamic controllability is PSPACE-complete.

## 1 Constraint Networks Under Conditional Uncertainty

*Constraint networks (CNs, [5])* are a framework to model *constraint satisfaction problems (CSPs)* and check the coherence of their relational constraints saying which combinations of values assigned to the variables are permitted. The main components of a constraint network are *variables*, *domains* and *constraints* and whenever all these components are under control we simply deal with a *consistency* problem asking us to find an assignment of values to all variables satisfying all constraints.

**Definition 1.** A *Constraint Network (CN)* is a tuple  $\langle \mathcal{X}, \mathcal{V}, D, \mathcal{C} \rangle$ , where  $\mathcal{X} = \{X_1, \dots, X_n\}$  is a finite set of variables,  $\mathcal{V} = \{v_1, \dots, v_m\}$  is a finite set of discrete values,  $D \subseteq \mathcal{X} \times \mathcal{V}$  is the *domain relation* (we write  $D(X) = \{v \mid (X, v) \in D\}$  to shorten the domain of  $X$ ) and  $\mathcal{C} = \{R_{S_1}, \dots, R_{S_k}\}$  is a finite set of relational constraints. Each  $R_{S_i}$  is defined over a *scope* of variables  $S_i \subseteq \mathcal{X}$  such that if  $S_i = \{X_{i_1}, \dots, X_{i_j}\}$ , then  $R_{S_i} \subseteq D(X_{i_1}) \times \dots \times D(X_{i_j})$ . A CN is consistent iff every variable  $X \in \mathcal{X}$  can be assigned a value  $v \in D(X)$  such that all constraints in  $\mathcal{C}$  are satisfied. Consistency of CNs is NP-complete [5].



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Fig. 1a shows an example of consistent CN, where a possible solution is  $X_1 = v_1$ ,  $X_2 = v_3$ ,  $X_3 = v_3$  and  $X_4 = v_2$ . Classic CNs do not address uncontrollable components. Indeed, when some component is out of control, satisfiability is, in general, not enough, and in such a case, we deal with a *controllability* problem. To address uncontrollable conditional constraints, CNCUs were proposed in [14, 16] as an extension of CNs to handle resource allocation problems under uncertainty in the context of business process management (BPM).

Let  $\mathcal{B} = \{a, b, \dots, z\}$  be a finite set of Boolean variables, a *label*  $\ell = \lambda_1 \dots \lambda_n$  is any finite conjunction of literals  $\lambda_i$  over the booleans in  $\mathcal{B}$ . The *empty label* is denoted by  $\square$ . The *label universe* of  $\mathcal{B}$ , denoted by  $\mathcal{B}^*$ , is the set of all possible (consistent) labels drawn from  $\mathcal{B}$ . For instance, if  $\mathcal{B} = \{a, b\}$ , then  $\mathcal{B}^* = \{\square, a, b, \neg a, \neg b, ab, a\neg b, \neg a b, \neg a \neg b\}$  (we omit the  $\wedge$  connective to ease reading). Two labels  $\ell_1, \ell_2 \in \mathcal{B}^*$  are *consistent* if and only if their conjunction  $\ell_1 \ell_2$  is satisfiable. A label  $\ell_1$  *entails* a label  $\ell_2$  (written  $\ell_1 \Rightarrow \ell_2$ ) if and only if all literals in  $\ell_2$  appear in  $\ell_1$  too (i.e., if  $\ell_1$  is more *specific* than  $\ell_2$ ).

**Definition 2.** A *Constraint Network Under Conditional Uncertainty (CNCU)* is a tuple  $\langle \mathcal{X}, \mathcal{V}, D, \mathcal{O}, \mathcal{B}, O, L, \preceq, \mathcal{C} \rangle$ , where:

- $\mathcal{X}, \mathcal{V}, D$  are the same as those given for CNs in Definition 1.
- $\mathcal{O} \subseteq \mathcal{X} = \{A?, B?, \dots\}$  is a set of *observation points*.
- $\mathcal{B} = \{a, b, \dots, z\}$  is a finite set of *booleans*.  $O: \mathcal{B} \rightarrow \mathcal{O}$  is a bijection assigning a unique observation point  $A?$  to each boolean  $a$ . When  $A?$  is assigned a value  $v \in D(A?)$ , the truth value of  $a$  is set by Nature and no longer changes.
- $L: \mathcal{X} \rightarrow \mathcal{B}^*$  is a mapping assigning a label  $\ell$  to each variable  $X$ .
- $\preceq \subseteq \mathcal{X} \times \mathcal{X}$  is a precedence relation on the variables. We write  $(X_1, X_2) \in \preceq$  (or  $X_1 \preceq X_2$ ) to express that  $X_1$  is executed *before*  $X_2$ .
- $\mathcal{C}$  is a finite set of *conditional constraints* of the form  $\ell \rightarrow R_S$ , where  $\ell \in \mathcal{B}^*$  and  $R_S$  is a classic relational constraint.

**Definition 3.** A CNCU is *well defined* iff all labels are consistent and:

1. For each  $X \in \mathcal{X}$ , if a literal  $a$  (or  $\neg a$ )  $\in L(X)$ , then  $L(X) \Rightarrow L(O(a))$  and  $O(a) \preceq X$ .
2. For each constraint  $(R_S, \ell) \in \mathcal{C}$ ,  $\ell \Rightarrow \bigwedge_{X \in S} L(X)$  and if a literal  $a$  (or  $\neg a$ )  $\in L(X)$ , then  $\ell \Rightarrow L(O(a))$ .
3. For each pair  $(X_1, X_2) \in \preceq$ ,  $L(X_1) \wedge L(X_2)$  is consistent.

Regarding the notions of well-definedness (initially proposed for conditional temporal networks in [6] and then adapted to CNCUs in [14, 16]), (1) and the second part of (2) say that any label must contain the labels of the observation points associated to each proposition embedded in each contained literal (label honesty). The first part of (2) says that a label on a constraint must be at least as expressive as any label in the scope of the relation (label coherence). (3) says that we cannot impose an order between two variables not taking part together in any execution.

Fig. 1b shows the graphical representation of a well-defined CNCU specifying 4 variables  $E?, X_1, X_2, X_3$ , where  $D(E?) = D(X_1) = D(X_3) = \{v_1, v_2\}$  and  $D(X_2) = \{v_2\}$ .  $E?$  is an observation point whose associated boolean is  $e$ . Order edges (directed thick edges) say that  $E?$  must be executed (i.e., assigned a value) before  $X_2$  and  $X_3$ , whereas  $X_1$  must be executed before  $X_2$ .  $E?, X_1$  and  $X_2$  are always executed as  $L(E?) = L(X_1) = L(X_2) = \square$  (empty label imposes no conditions).  $X_3$  is executed if and only if  $e$  is assigned true as  $L(X_3) = e$ , ignored otherwise. The CNCU specifies four constraints represented as labels on constraints edges (undirected thin edges). For example,  $(R, \square)$  between  $E?$  and  $X_1$  (see caption) represents a relation  $\square \rightarrow R$  saying that if  $E? = v_1$ , then  $X_1$  can be any value, whereas if  $E? = v_2$ , then  $X_1 = v_2$ . The constraint holds for any execution as its label is  $\square$ . Instead,  $(\neq, e)$  between  $E?$  and  $X_3$  says that if  $e$  is assigned true, then  $E? \neq X_3$ . Likewise, if  $e$  is assigned true, then  $X_1 \neq X_2$ , else  $X_1 = X_2$ .

A *scenario*  $s: \mathcal{B} \rightarrow \{\perp, \top\}$  is a total assignment of truth values to the booleans in  $\mathcal{B}$ . A scenario satisfies a label  $\ell$  (in symbols  $s \models \ell$ ) if  $\ell$  evaluates true under the interpretation given by  $s$ . Variables and



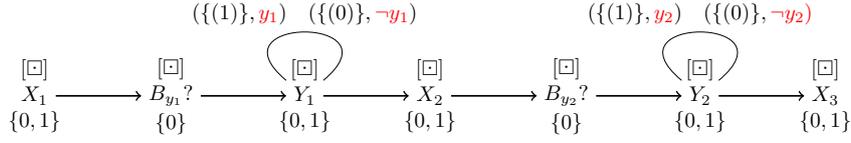


Figure 2: Example of reduction from  $\Phi \equiv \exists x_1 \forall y_1 \exists x_2 \forall y_2 \exists x_3 (x_1 \vee \neg y_2 \vee \neg y_1) \wedge (x_2 \vee \neg x_3 \vee y_1)$ . We encode the first clause as  $\square \rightarrow R_1$  and  $R_1 \equiv D(X_1) \times D(Y_2) \times D(Y_1) \setminus \{(0, 1, 1)\}$  and the second one as  $\square \rightarrow R_2$  where  $R_2 \equiv D(X_2) \times D(X_3) \times D(Y_1) \setminus \{(0, 1, 0)\}$ . Being ternary,  $\square \rightarrow R_1$  and  $\square \rightarrow R_2$  are not shown.

constraints to connect each previous discussed “gadget” encoding a quantified variable of  $\Phi$  to the next one according to the order in which these variables appear in the quantified part of  $\Phi$ . Finally, for each clause  $C_i$  we add a relational constraint  $\square \rightarrow R_{S_i}$  such that the scope  $S_i$  contains the three variables embedded in the literals appearing in  $C_i$ , whereas the set of tuples is the cross product of the domains of such variables minus the unique tuple falsifying the clause (each of these relations has exactly  $2^3 - 1$  tuples). Now it is easy to see that the reduction runs in polynomial time. Fig. 2 provides an example.  $\square$

**Theorem 1.** *Weak controllability of CNCUs is  $\Pi_2^P$ -complete.*

*Proof. Hardness:* Lemma 1 allows us to reduce in polynomial time a QBF having the form  $\forall y_1, \dots, \forall y_n, \exists x_1, \dots, \exists x_m \varphi$ . Solving such an instance of QBF is known to be  $\Pi_2^P$ -complete. The reduction constructs a CNCU which is weakly controllable iff  $\Phi$  is satisfiable. **Membership:** Any disqualification consists of a specific scenario  $s$  for which there is no solution to the corresponding projection. Once we have  $s$ , the inconsistency of the projection can be verified in non-deterministic polynomial time by using topological sort plus any decision procedure to check consistency of CNs.  $\square$

**Theorem 2.** *Strong controllability of CNCUs is NP-complete.*

*Proof.* We already discuss that a CNCU is strongly controllable if it admits a total order and the CN *super-projection* is consistent. Such a projection is computable in polynomial time (Algorithm 1, line 1).

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**Algorithm 1:** CncuSC( $\mathcal{N}$ )

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**Input:** A CNCU  $\mathcal{N} = \langle \mathcal{X}, \mathcal{V}, D, \mathcal{O}, \mathcal{B}, O, L, \preceq, \mathcal{C} \rangle$

**Output:** Yes, if  $\mathcal{N}$  is strongly controllable. No otherwise.

1  $\mathcal{N}^* = \langle \mathcal{X}, \mathcal{V}, D, \{R_S \mid \ell \rightarrow R_S \in \mathcal{C}\} \rangle$

$\triangleright$  Compute super-projection

2 **if** no total order on  $\mathcal{X}$  meets  $\preceq$  **then return** No

$\triangleright$  Topological Sort

3 **return** CN-Consistency( $\mathcal{N}$ )

$\triangleright$  Any consistency decision procedure for CNs

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We compute a total order by means of a run of Topological Sort. If none exists, the CNCU is not strongly controllable. A strong assignment for a CNCU is an assignment satisfying all constraints regardless of their labels. We claim that there exists a strong assignment for  $\mathcal{N}$  iff there exists an assignment for the corresponding super-projection  $\mathcal{N}^*$ . The former because a strong solution satisfying all constraints of  $\mathcal{N}$  satisfies trivially their intersection in  $\mathcal{N}^*$ , the latter because a solution to  $\mathcal{N}^*$  is a solution satisfying the intersection of all constraints of  $\mathcal{N}$ , therefore, a solution satisfying all constraints of  $\mathcal{N}$ . Therefore, strong controllability is equivalent to CN-consistency which is NP-complete.  $\square$

**Theorem 3.** *Dynamic controllability of CNCUs is PSPACE-complete.*

*Proof. Hardness:* Once again, Lemma 1 allows us to reduce in polynomial time a QBF without restrictions on the number of alternations of the quantifiers to a CNCU that is dynamically controllable iff  $\Phi$  is satisfiable. **Membership:** Algorithm 2 is a polynomial space algorithm to decide dynamic controllability of any CNCU. In Algorithm 2,  $s$  is a scenario,  $\alpha$  an *assignment*  $\alpha: \mathcal{X} \rightarrow \mathcal{V}$  of values to variables,  $\omega$  is a permutation  $(X_{i_1}, \dots, X_{i_n})$  of  $\mathcal{X}$  (with  $n = |\mathcal{X}|$ ). We write  $\omega(X_{i_j})$  to refer to the index  $j$  of  $X_{i_j}$  in  $\omega$ . Given a CNCU  $\mathcal{N} = \langle \mathcal{X}, \mathcal{V}, D, \mathcal{O}, \mathcal{B}, O, L, \preceq, \mathcal{C} \rangle$ , and a triple  $(s, \alpha, \omega)$  we say that  $(s, \alpha, \omega) \models \mathcal{N}$  iff:

1. for each  $X \in \mathcal{X}$ ,  $\alpha(X) \in D(X)$
2. for each  $\ell \rightarrow R_{\{X_{i_1}, \dots, X_{i_n}\}} \in \mathcal{C}$ , if  $s \models \ell$ , then  $(\alpha(X_{i_1}), \dots, \alpha(X_{i_n})) \in R_{\{X_{i_1}, \dots, X_{i_n}\}}$

3. for each  $(X, Y) \in \preceq$ ,  $\omega(X) < \omega(Y)$ .

□

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**Algorithm 2:** CncuDC( $\mathcal{N}$ )

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**Input:** A CNCU  $\mathcal{N} = \langle \mathcal{X}, \mathcal{V}, D, \mathcal{O}, \mathcal{B}, O, L, \preceq, C \rangle$

**Output:** Yes, if  $\mathcal{N}$  is dynamically controllable. No otherwise.

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1 CncuDC ( $\mathcal{N}$ )
2   Let  $s, \alpha, \omega$  be an empty scenario, assignment and ordering.
3   return Explore( $\mathcal{N}, \mathcal{X}, s, \alpha, \omega$ )

4 Explore ( $\mathcal{N}, \mathcal{X}, s, \alpha, \omega$ )
5   if  $\mathcal{X} = \emptyset$  then return  $(s, \alpha, \omega) \models \mathcal{N}$                                 ▷ leaf check
6   for  $X \in \mathcal{X}$  do                                                                    ▷ pick a variable
7      $\omega' \leftarrow (\omega, X)$                                                        ▷ append  $X$  to the current ordering
8     for  $v \in D(X)$  do                                                                ▷ look for a value to assign to  $X$ 
9        $\alpha' \leftarrow \alpha \cup \{ \alpha'(X) \leftarrow v \}$                        ▷ extend current plan
10      if  $X \in \mathcal{O}$  then                                                                ▷ case 1
11        Let  $x$  be the boolean associated to  $X$ 
12         $s' \leftarrow s \cup \{ s(x) \leftarrow \top \}$                                ▷ extend scenario (positive case)
13         $s'' \leftarrow s \cup \{ s(x) \leftarrow \perp \}$                              ▷ extend scenario (negative case)
14        if Explore( $\mathcal{N}, \mathcal{X} \setminus \{X\}, s', \alpha', \omega'$ )  $\wedge$  Explore( $\mathcal{N}, \mathcal{X} \setminus \{X\}, s'', \alpha', \omega'$ ) then
15          return Yes
16      if  $X \notin \mathcal{O} \wedge$  Explore( $\mathcal{N}, \mathcal{X} \setminus \{X\}, s, \alpha', \omega'$ ) then return Yes  ▷ case 2
17   return No                                                                           ▷ “no strategy” from subtree

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### 3 Conclusions and Future Work

We discussed the computational complexity of weak, strong and dynamic controllability of CNCUs. Weak controllability is  $\Pi_2^P$ -complete, strong controllability is NP-complete (as it is equivalent to a satisfiability problem), whereas dynamic controllability is PSPACE-complete.

As future work we plan to compare with complexity results for other classes of (temporal)-constraint networks such as those discussed (or employed) in [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 15, 14, 16, 17].

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