Definability problem and automata for first order logic over trees

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joint work with M. Benerecetti, D. Della Monica, F. Mogavero and G. Puppis

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Many generalizations: infinite words, infinite trees, even **graphs of bounded treewidth**!

Monadic Second Order Logic

Monadic Second Order Logic (MSO) is Second Order Logic in which second order quantification is limited to monadic predicates, i.e., **sets.**

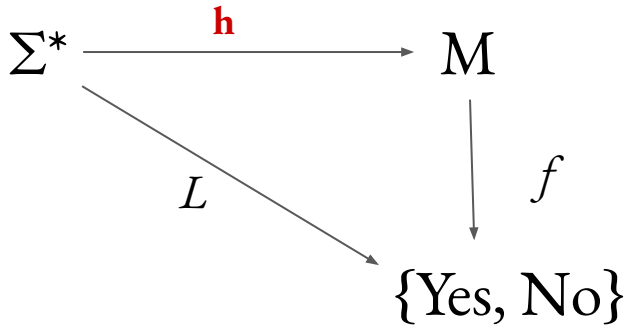
Over words, MSO can quantify over **sets of "positions"** of the word.
Over trees, it can quantify over **sets of nodes** of the tree.

Another notion of regularity (for words)

A finite **monoid M** consists of a finite set S equipped with a binary multiplication operation, which is associative, and a neutral element.

A monoid **homomorphism** is a function between underlying sets of monoids, that preserves the multiplication operation.

A language $L \subseteq \Sigma^*$ is **recognized** by a monoid homomorphism iff

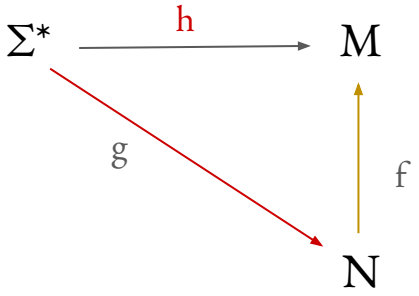


where M is a finite monoid and *f* is some function that specifies the "accepting set" of M

Another notion of regularity (for words)

Theorem
A language of finite words L over a finite alphabet Σ is regular iff it is recognized by a finite monoid homomorphism.

The **syntactic monoid** of a language L is the target of a **surjective** monoid homomorphism h, such that it recognizes L and the following commutes:



For every surjective homomorphism that recognises L, there is a unique surjective homomorphism that extends it to h.

Every regular language has an **effectively presentable** syntactic monoid.

A natural question

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Given a fragment F of MSO, is the following decidable?

Input: A regular language L
Question: Is L definable in F?

Positive answers

Over finite words, many positive answers.

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Theorem (Pin & Weil (1997))

FO[<] with two variables = DA monoids ($(ab)^\lambda = (ab)^\lambda a(ab)^\lambda$)

Why is this effective?

The previous characterisations all yielded **algorithms** to decide the definability in the given logic. The algorithms works this way:

1. Take an arbitrary representation of the language (regular expressions, MSO, automata, ...).
2. Transform the representation of the language in a finite monoid.
3. Compute the syntactic monoid of the language.
4. Check if the syntactic monoid satisfies the equation specified in the theorem.

What about tree languages?

How can one approach this problem for **trees**?

First definitions and approaches: Thomas (1984, 1987).

In the following, finite and infinite trees will be mixed without a lot of precision.

Logics for trees

Thomas identifies three “fragment” of MSO over trees:

- Monadic **Antichain** Logic: set quantification limited to **incomparable nodes** wrt <.
- Monadic **Chain** Logic: set quantification limited to **comparable nodes** wrt **<.**
- FO/Monadic **Path** logic: quantification limited to nodes/paths.

Aperiodicity for trees

A **context** is a tree with exactly **one** special constant instead of a leaf (or of an infinite subtree, if we are in the case of infinite trees).

Given a finite alphabet Σ, the set S_Σ of all the contexts obtainable from Σ is a monoid with "**tree concatenation**" as the multiplication operation.

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A tree language T is **aperiodic** if for every element a of its syntactic monoid it holds $a^\lambda = a^{\lambda+1}$.

Monadic Antichain Logic = MSO

The definability problem for Antichain Logic is **trivial**, because Potthoff & Thomas (1993) show that

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over trees.

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2. There are many different **modal logics** effectively equivalent to it.
3. There are two **automaton-based characterisations** (this work).

Again, no clue about an algebraic characterisations, not even a conjecture.

Side problem

Chain Logic and FO are very close. Can one at least decide if a
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Solving this problem would reduce the FO-definability problem to the Chain Logic-definability.

Other directions

Many other algebraic frameworks were pursued after Thomas. The two most important ones are:

1. **preclones** (Esik & Weil, 2005)
2. **forest algebras** (Bojanczyk & Walukiewicz, 2008)

In both these formalisms (2009, 2012) a **non-effective** characterisation of FO-definability is obtained. Again, no insight about an **effective** characterisation.

Positive answers

There are some positive results. For the following logics, the definability problem over trees is decidable:

- temporal logics EX, EF, EX+EF, EF + $F^{\leftarrow}$.
- Boolean combinations of Σ_1 formulas.
- Δ_2.
- FO[S_1,...S_k].
- FO[<] with two variables.

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Is there **another way** to gain a decent understanding of a language class?

What about automata?

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Recent advances: **automata for Monadic Path Logic and Monadic Chain Logic** (Bozzelli, Benerecetti, Mogavero & Peron, 2024).

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1. M. Bojanczyk (2004) proved the equivalence over **finite binary trees** of FO with **successors** and order to a **cascade product** of a specific deterministic automaton.
2. C. Ford (2019) introduced a class of automaton (let's call it FordAuto) translatable into FO with order. Did not prove the other direction.

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Main result:

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1. we investigated two **branching-time temporal logics** already shown equivalent to FO over infinite trees;
2. we found **two automaton classes** effectively equivalent to the temporal logics, and consequently to FO;
3. from the automaton-based characterisations, we got **normal forms** for the temporal logics and gained insight into **the behaviour of FO over infinite trees**.

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- Unordered: the order of the successors of a given node is irrelevant.

First Order Logic over infinite trees

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What FO can say	What FO can't say
There are at least three successors of a given node that share the property *P*.	The third, fifth and eighth successor of a given node share the property *P*.

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$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid D^{n}\varphi \mid EX\varphi \mid E\varphi U\varphi \mid \sout{E\varphi R\varphi} \mid Y\varphi \mid \varphi S\varphi$$

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What Polarized CTL can't say
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Polarized CTL is a proper fragment of PDL.

Open problem: what is the fragment of μ-calculus coinciding with Polarized CTL?

CTL* over finite paths

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- the usual path quantifiers refer only to non-empty **finite paths;**
- it is **equivalent to FO** over trees, too.

$$\varphi := p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathrm{D}^{n}\varphi \mid \mathrm{E}\psi$$
$$\psi := \varphi \mid \neg\psi \mid \psi \vee \psi \mid \mathrm{X}\psi \mid \psi \mathrm{U} \psi$$

Finite path quantification

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The semantic restriction of CTL* over finite paths **trivializes many formulas.** The simplest examples are:

$$\mathrm{E}\dot{\mathrm{X}}\varphi \leftrightarrow \top \qquad\qquad \mathrm{A}\mathrm{X}\,\varphi \leftrightarrow \bot$$

where $\dot{\mathrm{X}}$ is the **weak version** of the X operator.

Automaton-based characterisations

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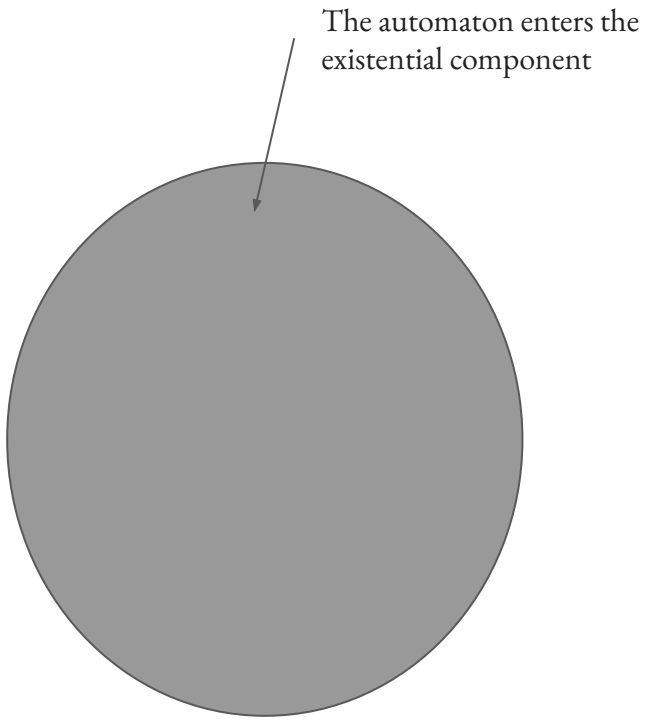
- Alternating: they mix nondeterminism and universal branching;
- Weak: they can switch a finite number of times between accepting and rejecting states;
- Hesitant: the state-set is partitioned in a way that simulates the difference between existential and universal path quantifiers.

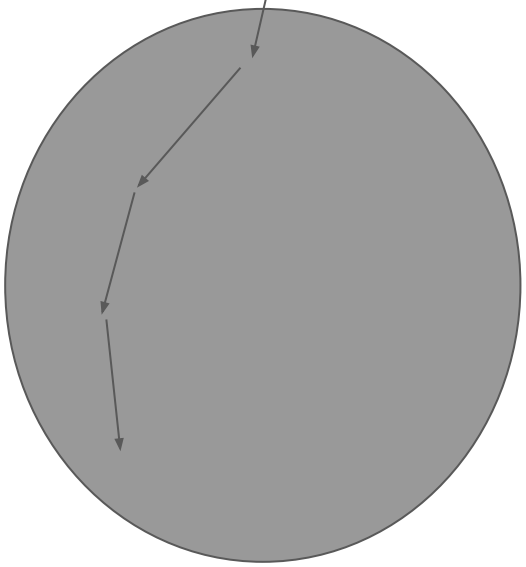
Hesitant restriction: Existential components



Hesitant restriction: Existential components

The automaton enters the existential component

The automaton always picks **exactly one** successor in the component and send it to one child of the current input.



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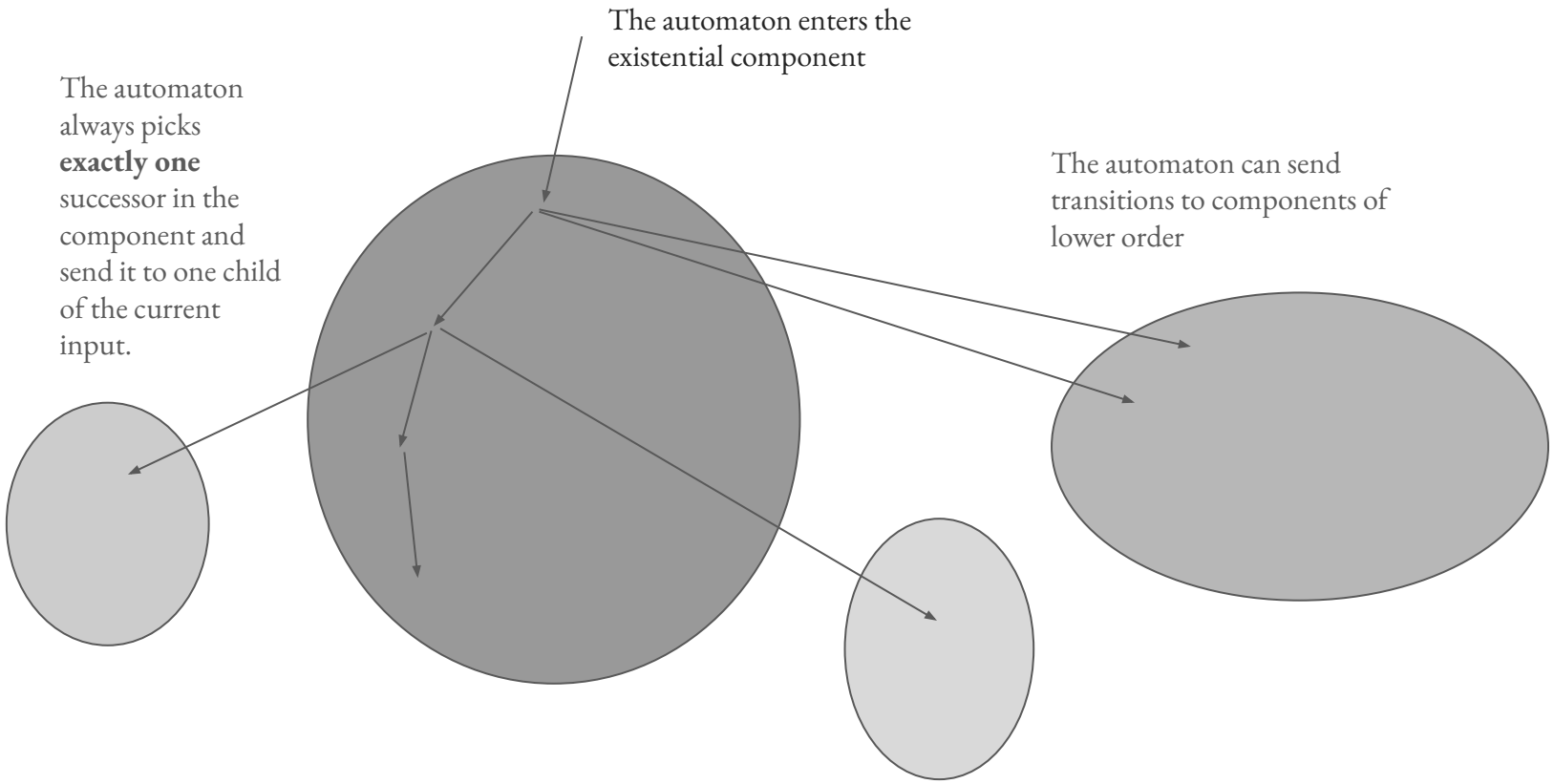
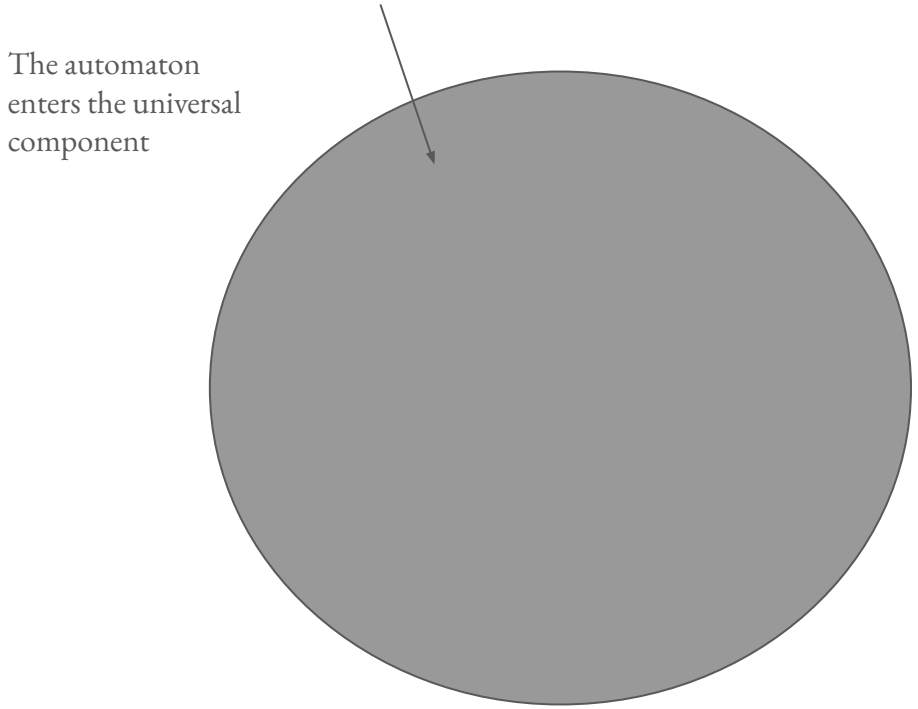


Hesitant restriction: Universal components



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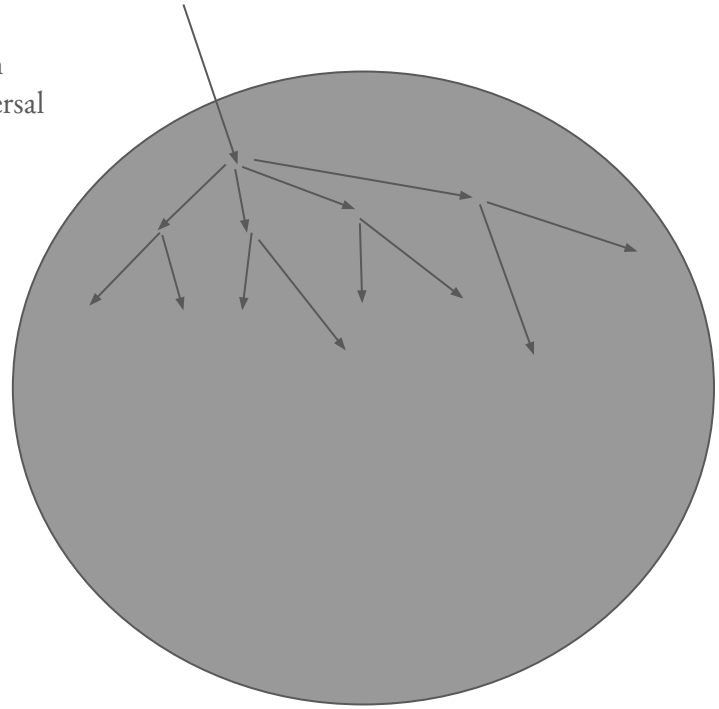
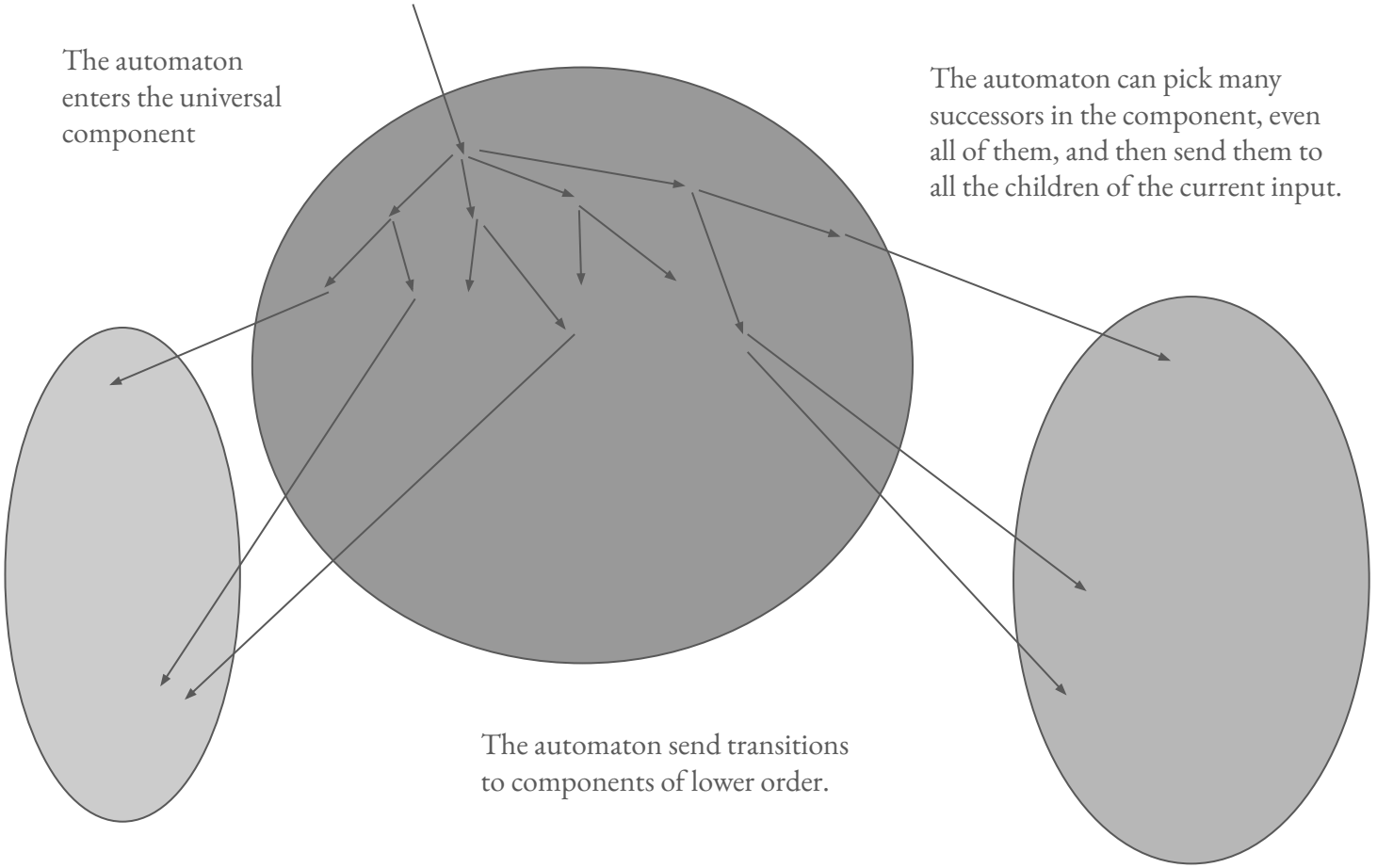


The automaton can pick many successors in the component, even all of them, and then send them to all the children of the current input.

Hesitant restriction: Universal components



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One has to impose different restrictions on the state set to obtain equivalence with FO.

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Similar to alternating automata for **FO over words.**

Conjecture

Call the automaton-class modeled after Polarized CTL: PolarizedAuto. We proved:

Theorem
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We conjecture that the FordAuto is **strictly less expressive** than FO, because of the following.

Conjecture
FordAuto is equivalent to 1-way PolarizedAuto.

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Counter-free automata are the automata equivalent to **FO over words**.

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SafetyLTL = Safety fragment of FO over words.

CoSafetyLTL = CoSafety fragment of FO over words.

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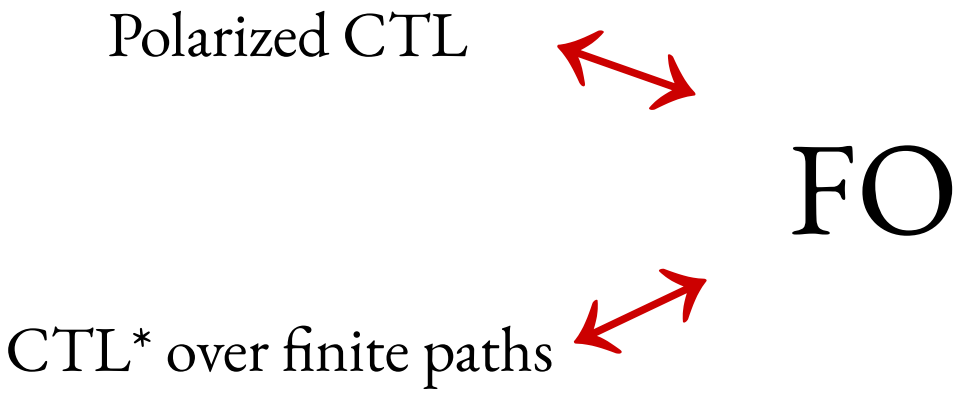

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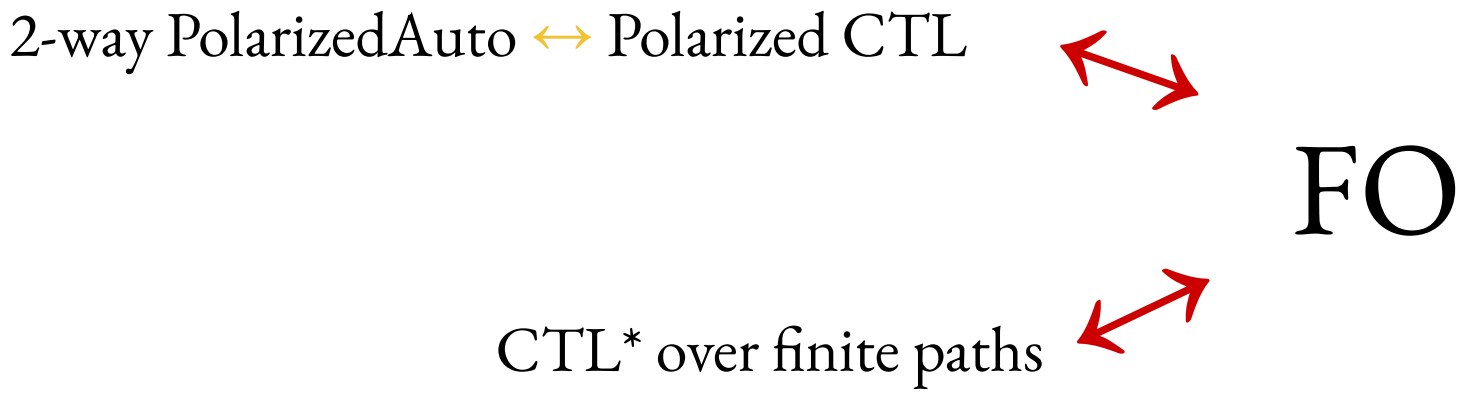

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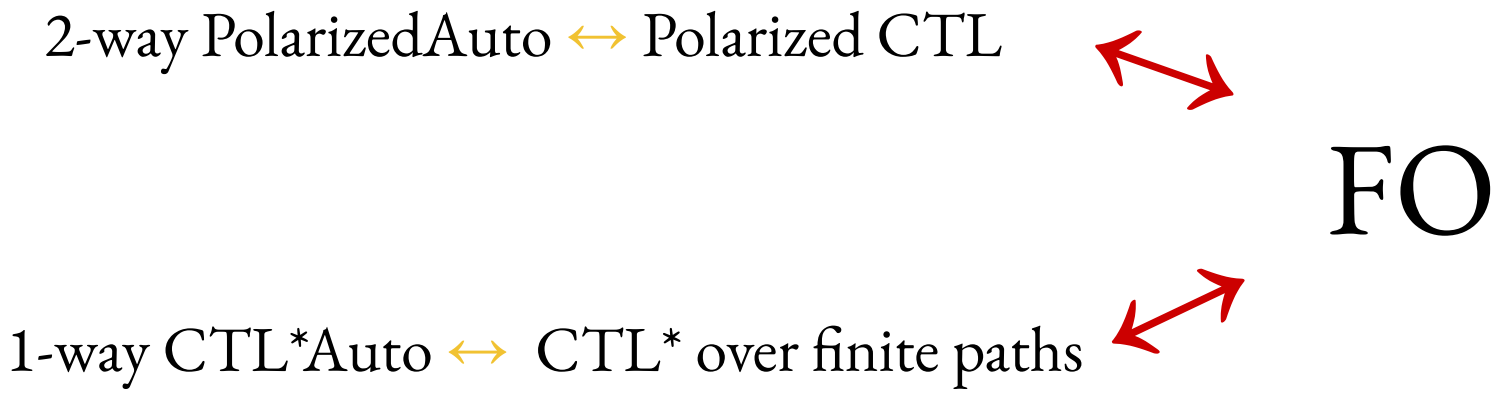
Automaton-class modeled after CTL* over finite paths: CTL*Auto.



Normal forms and insight into FO over infinite trees

The two automaton-based characterisations of FO we obtained offered as a by-product two **normal forms** for Polarized CTL:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \bigvee \varphi \mid \mathrm{D}^n\varphi \mid \mathrm{EF}\psi$$
$$\psi ::= \varphi \mid \psi \bigwedge \psi \mid \psi \bigvee \psi \mid \mathrm{Y}\psi \mid \psi\mathrm{S}\psi$$

and CTL* over finite paths:

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Take-away messagge
Over infinite trees, when FO predicates existentially over a path, it can only express **co-safety properties**, while when it predicates over all paths, it can only express **safety properties**.

Conclusions

We obtained:

- two **automaton-based characterisations** for FO over infinite trees
- **normal forms** for two temporal logics equivalent to FO
- **insight** into the behaviour of FO over infinite trees

Open problems: Definability problem

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is **decidable.**

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Very similar to normal form of Polarized CTL!

Idea: Investigate definability problem of Polarized CTL over **finite** trees.

Open problems: Automata simulation

2-way PolarizedAuto and 1-way CTL*Auto are equivalent by the **equivalence of temporal logics**.

Interesting problem: Find **direct simulation** of each automaton by the other.

Open problems: Full CTL with past expressiveness

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Polarized CTL with past < Full CTL with past < CTL*

Thank you
for your attention!